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In recent years there has been considerable experimental[1-5] and some theoretical[6,7] interest in the spatial-temporal growth of instabilities arising when a beam is injected into a plasma. There does not appear, however, to be any theoretical treatment of the modification of the growth of an initial perturbation due to the presence of a spatial boundary. This modification will be illustrated for the case of a monoenergetic electron beam injected into a cold plasma with an infinitely massive ion background.

The treatment is based on the linearized hydrodynamic equations and Maxwell's equations for the normalized densities, the drift velocities, and the electric field, ($\mathcal{E}$ = (e/m) E):

$$n_{P,t} + u_{P,x} = 0 \tag{1}$$

$$u_{P,t} + \mathcal{E} = -\nu u_P \tag{2}$$

$$n_{B,t} + u_o n_{B,x} + u_{B,x} = 0 \tag{3}$$

$$u_{B,t} + u_o u_{B,x} + \mathcal{E} = 0 \tag{4}$$

$$\mathcal{E}_{,t} - \omega_P^2 u_P - \omega_B^2 (u_B + u_o n_B) = 0 \tag{5}$$

Eq. (2) contains a phenomenological collision term, representing collisions between the plasma electrons and the ion background. The remaining Maxwell's equation:

$$\mathcal{E}_{,x} + \omega_P^2 n_P + \omega_B^2 n_B = 0 \tag{6}$$

is treated as a condition to be satisfied at t = 0, whence it is satisfied for all later times.

Eqs. (1-5), which are of the following form (repeated indices are summed):

$$a_{ij}f_{j,t} + b_{ij}f_{j,x} + c_{ij}f_j = 0 \qquad i,j = 1,\ldots,5 \tag{7}$$

are to be solved for $x > 0$ and $t > 0$ subject to conditions: $f_i(0,t) = g_i(t)$ and $f_i(x,0) = h_i(x)$. These conditions are not all independent since Eqs. (2) and (5) must be satisfied at $x = 0$, and equation (6) must be satisfied at $t = 0$. The first two of these restrictions are due to the fact that characteristics of Eqs. (1-5) lie on the boundary $x = 0$.

The double Laplace transformation solution to Eqs. (1-5) is:

$$f_i(x,t) = (2\pi i)^{-2}\int dp dq\ e^{pt + qx}\ F_i(p,q) \tag{8}$$

$$F_i(p,q) = [a_{jk}H_k(q) + b_{jk}G_k(p)]\ d_{ji}(p,q)/D(p,q) \tag{9}$$

H_i and G_i are the one dimensional Laplace transforms of $h_i(x)$ and $g_i(t)$. D is the determinant of the matrix $a_{ij}p + b_{ij}q + c_{ij}$, and d_{ij} is the cofactor. ($D = 0$ is the dispersion relation, apart from a multiplicative factor p, for the infinite case.) The integration paths in (8) are to the right of all singularities of the integrand.

If the transforms H_k contain only simple poles at $q = \alpha_i$, $i = 1\ldots$, the q integration in Eq. (8) may be performed to give:

$$f_i(x,t) = \frac{1}{2\pi i}\sum_{\ell=1,2}\int dp\ e^{pt + q_\ell x}\ \frac{[a_{jk}H_k(q_\ell) + b_{jk}G_k(p)]\ d_{ji}(p,q_\ell)}{(\partial D(p,q)/\partial q)\ _{q=q_\ell}}$$

$$+ \frac{1}{2\pi i}\sum_m\int dp\ e^{pt + \alpha_m x}\ \frac{a_{jk}d_{ji}(p,\alpha_m)}{D(p,\alpha_m)}\ \text{Residue}\ (H_k(\alpha_m)) \tag{10}$$

where $q_\ell(p)$ are the roots of $D(p,q(p)) = 0$:

$$q_{1,2}(p) = \frac{-p}{u_o} \pm \frac{i\omega_B}{u_o}\left[\frac{p(p + \nu)}{p(p + \nu) + \omega_p^2}\right]^{\frac{1}{2}}$$

If $x > u_o t$, the first term in Eq. (10) does not contribute, since the contour may be closed at infinity in the right half-plane, where it encloses no singularities. (This result is independent of whether the H_k have only simple poles.) The expression for $f_i(x,t)$ is then independent of the presence of the boundary and consists, as in the unbounded case, of plane wave terms proportional to exp $(p_\ell(\alpha_i)t + \alpha_i x)$ where $p_\ell(q)$, $\ell = 1,\ldots,4$ are the roots of $D(p(q),q) = 0$.

If $x < u_o t$, the first term of Eq. (10) has singularities at: (a) singularities of $q_\ell(p)$, (b) singularities of $H_k(q_\ell(p))$ apart from those in (a), and (c) singularities of $G_k(p)$, which we assume to be simple poles at $p = \beta_i$, $i = 1,\ldots$. The contour can, in this case, be closed in the left half-plane, and it can be shown that the singularities (b) exactly cancel the second term in Eq. (10). This cancellation eliminates the plane wave terms which appear for the infinite plasma. In their place the singularities (a), which include the essential singularities at $p = -\frac{1}{2}\nu \pm i\,[\omega_p{}^2 - \frac{1}{4}\nu^2]^{\frac{1}{2}}$, give the more complicated dependence due to the presence of the boundary. The singularities (c) give rise to plane wave terms proportional to $\exp(\beta_i t + q_\ell(\beta_i)x)$.

To illustrate the long time behavior of an initial perturbation, consider the case: $n_p(x,0) = N_o$ sinkx and $\mathcal{E}(x,0) = -N_o\omega_p{}^2 k^{-1}(1-\cos kx)$. All other boundary and initial conditions are taken to be zero. The integral (10) around the essential singularity can be evaluated by the saddle point method in the limit of large t. It can be shown that this method is valid in the present case, and that only two of the eight saddle points which occur contribute to the asymptotic behavior. The result is:

$$\mathcal{E}(x,t) = N_o k u_o^2 \, (6\pi)^{-\frac{1}{2}} \, (\omega_p/\omega_B)^2 \, \beta^{\frac{1}{2}} \chi^{1/3} \, \tau^{-5/6}$$

$$\times \exp\left[\tfrac{3}{4}\sqrt{3}\,\beta\chi^{2/3}\,\tau^{1/3} - (\tfrac{1}{2}\nu/\omega_p)\,\tau\right]$$

$$\times \sin\left[\beta^{-3}\tau - \frac{3}{4}\,\beta\chi^{2/3}\tau^{1/3} - \phi - \frac{1}{12}\,\pi\right] \qquad (12)$$

where $\beta = [1 - (\frac{1}{2}\nu/\omega_p)^2]^{-1/6}$, $\chi = \omega_B x/u_o$, $\tau = \omega_p t$, and $\sin\phi = \frac{1}{2}\nu/\omega_p$. This result is valid provided $x \ll u_o t$ and $\beta\chi^{2/3}\tau^{-2/3} \ll 1 \ll \beta\chi^{2/3}\tau^{1/3}$. It should be noted that the exponential factor, which dominates the form of $\mathcal{E}(x,t)$, is independent of the initial disturbance.

The above results yield the following picture for the development of an initial disturbance. If the disturbance is unstable according to the usual theory for spatially unbounded systems, it grows as $e^{\gamma t}$ only until $t = x/u_o$. For $t \gg x/u_o$ the amplitude goes as in Eq. (12). It reaches a maximum at $t_o = \omega_p^{\frac{1}{2}}(\frac{1}{2}\sqrt{3}\,\beta/\nu)^{3/2}\,(\omega_B x/u_o)$ and finally decays as $e^{-\frac{1}{2}\nu t}$. If the collision frequency is large, t_o does not occur in the asymptotic region $t \gg x/u_o$, so the behavior is always $e^{-\frac{1}{2}\nu t}$ in this region. On the other hand if $\nu = 0$, the disturbance grows indefinitely, and one has essentially an absolute rather than a convective instability. This is due to the fact that there are characteristics of Eqs. (1-5) which are parallel to the t axis.[8] The presence of collisions thereby insures the validity of the linear approximation for all times, provided x is not too large.

In experimental measurements what is observed is the result of many disturbances originating at different prior times. Since each disturbance has a limited lifetime (for any fixed x and $\nu \neq 0$), the cumulative effect is a stationary state which varies with x. In the present situation the lifetime depends critically on the collision frequency, so one expects the level of fluctuations in the stationary state to also be strongly influenced by ν. While a rough estimate of this stationary state can be obtained from the above results, a more accurate analysis is desirable. Such an analysis is presently being performed.

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